

# On The Phase Transition in D=3 Yang-Mills Chern-Simons Gauge Theory

John M. Cornwall\*

*Physics Department, University of California at Los Angeles  
405 S. Hilgard Ave., Los Angeles Ca 90095-1547*

## Abstract

$SU(N)$  Yang-Mills theory in three dimensions, with a Chern-Simons term of level  $k$  (an integer) added, has two dimensionful coupling constants,  $g^2k$  and  $g^2N$ ; its possible phases depend on the size of  $k$  relative to  $N$ . For  $k \gg N$ , this theory approaches topological Chern-Simons theory with no Yang-Mills term, and expectation values of multiple Wilson loops yield Jones polynomials, as Witten has shown; it can be treated semiclassically. For  $k = 0$ , the theory is badly infrared singular in perturbation theory, a non-perturbative mass and subsequent quantum solitons are generated, and Wilson loops show an area law. We argue that there is a phase transition between these two behaviors at a critical value of  $k$ , called  $k_c$ , with  $k_c/N \approx 2 \pm .7$ . Three lines of evidence are given: First, a gauge-invariant one-loop calculation shows that the perturbative theory has tachyonic problems if  $k \leq 29N/12$ . The theory becomes sensible only if there is an additional dynamic source of gauge-boson mass, just as in the  $k = 0$  case. Second, we study in a rough approximation the free energy and show that for  $k \leq k_c$  there is a non-trivial vacuum condensate driven by soliton entropy and driving a gauge-boson dynamical mass  $M$ , while both the condensate and  $M$  vanish for  $k \geq k_c$ . Third, we study possible quantum solitons stemming from an effective action having both a Chern-Simons mass  $m$  and a (gauge-invariant) dynamical mass  $M$ . We show that if  $M \gtrsim 0.5m$ , there are finite-action quantum sphalerons, while none survive in the classical limit  $M = 0$ , as shown earlier by D'Hoker and Vinet. There are also quantum topological vortices smoothly vanishing as  $M \rightarrow 0$ .

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\*Electronic address: cornwall@physics.ucla.edu

# 1 Introduction

We study here a three-dimensional Euclidean Yang-Mills theory with a Chern-Simons mass term added[1], or just YMCS theory for short. As is well-known, the Chern-Simons term generates a gauge-boson mass  $m$  of magnitude  $kg^2/4\pi$ , where  $k$  is an integer (which we can and will choose to be non-negative). Although a massless pole still remains in the gauge-boson propagator, this pole is associated with kinematical factors which remove all infrared divergences, at least in gauge-invariant quantities. It may therefore seem, at a naive glance, that YMCS has a perfectly respectable perturbation theory, at least in the sense that every term in the series is well-defined, even if the series might not converge. Of course, one might expect perturbation theory to fail for small  $k$ , because (see below) one expansion parameter of YMCS is  $N/k$  (modulo a numerical factor) and becomes large for  $k \ll N$ . But there is more to it than just the size of this parameter; there is a critical minus sign which is analogous to the sign of the  $\beta$ -function in  $d = 4$  gauge theory, which drives the phenomena we discuss. In the eighties a number of authors calculated some one-loop terms in the perturbation series[1, 2] for YMCS, especially for the conventionally-defined Feynman propagator. Unfortunately, because of the gauge dependence of the conventionally-defined self-energy, it was not possible to see this sign structure gauge-invariantly from these calculations. But we will show in a gauge-invariant way that YMCS theory has a tachyonic problem in physical amplitudes, in spite of the fact that the Euclidean Chern-Simons term has a factor of  $i$  which makes the mass  $m$  non-tachyonic. This tachyonic problem occurs for the same reason (gluon spin couplings) that there is an infrared renormalon pole in the running charge in  $d = 4$  gauge theory, as we will elaborate below. In  $d = 4$  one is used to associating this tachyonic pole with asymptotic freedom, and we will use this same phrase as a shorthand for the sign structure of YMCS, even though as a  $d = 3$  theory it is superrenormalizable and has no renormalization group.

The essential point is that when  $k$  is less than a critical value  $k_c$ , of  $O(N)$ , the mass  $m$  is too small to overcome the tachyonic tendencies associated with what would be called asymptotic freedom in  $d = 4$ . Then the situation is analogous to the  $k = 0$  case, which is clearly non-perturbative[3, 4, 5, 6]. The cited authors discuss the need for generation of a non-perturbative dynamical gauge mass  $M$  of  $O(Ng^2)$ , to cure the infrared divergences of perturbation

theory. Dynamical mass generation is driven by (and drives) the formation of a gauge-boson condensate, as reflected in a positive value of  $\langle (G_{ij}^a)^2 \rangle$ . In spite of being  $O(g^2)$ , the dynamical mass vanishes to all orders of perturbation theory, since there is no acceptable mass counterterm to put in the action. Although the local gauge symmetry is exactly preserved in the course of dynamical mass generation, the massive gauge bosons are necessarily accompanied by massless excitations which, like Goldstone bosons in symmetry breaking, are “eaten” and do not appear in the  $S$ -matrix as massless poles. They do have profound physical effects, however. The massless excitations correspond to long-range pure-gauge parts of the gauge potential, which have topological significance and generate, among other things, the area law for Wilson loops[7, 8, 9]. The confinement (or area-law) mechanism, just as in lattice gauge theory[10], is one of topological linkage of a condensate of closed vortices (which can be cut open by the formation of a monopole-antimonopole pair) with the Wilson loop. In fact, the area law is described via the standard Gauss linking integral for two closed strings. Moreover, the fluctuations of the Chern-Simons term are also described[8, 9, 11] as averages over powers of the linking number, in this case of vortex strings with themselves or other vortex strings in the vacuum. These vortices are one of several types of solitons which occur in the dynamically-massive theory (another is the sphaleron corresponding to no Higgs symmetry breaking[12, 13]) but not in the classical massless theory; that is, these solitons owe their existence to a dynamical mass. Since this mass is purely a non-perturbative quantum effect, we call these quantum solitons. The entropy of the vortex solitons is larger than their free energy, and a condensate of the vortices forms, which is responsible for dynamical mass generation (for example, Lavelle [14] has shown that the dynamical gauge-boson mass behaves rather like a constituent quark mass, with the squared mass vanishing at large momentum  $p$  at a rate involving the relevant condensate expectation value:  $\langle g^2(G_{ij}^a)^2 \rangle/p^2$ ).

We will show that in the opposite case  $k > k_c$ , the tachyonic problem and its cures, namely generation of a dynamical mass and condensate, go away and the theory smoothly merges, as  $k \rightarrow \infty$ , into Witten’s[29, 27] topological theory which is accessible by semiclassical means. Because the dynamical mass and condensate vanish, there are no quantum solitons, and it is known that there are no classical solitons of finite action in YMCS theory[17].

The purpose of the present work is to describe the nature of this unusual phase transition and to make some (probably not very accurate) estimates of

some of the numbers involved, such as the critical value  $k_c$  of  $k$ . The essence of this phase transition is the analog[5, 6, 18] of asymptotic freedom in four dimensions, as indicated by a crucial sign coming from the spin-dependent gauge-boson couplings[19]. Of course, there is no renormalization group in  $d = 3$  gauge theory, so the usual calculation of a  $\beta$ -function which reveals this sign structure in  $d = 4$  cannot be done. But it is possible to calculate a running charge[5, 6, 18] using the pinch technique, which identifies[5, 20] a gauge-invariant gauge-boson proper self-energy from the  $S$ -matrix, and the square of this running charge becomes negative for sufficiently small momentum, because of a tachyonic (real Euclidean) zero in a gauge-invariant proper self-energy. If this zero were missing, many or all of the effects we find would not occur.

We can give a simple quantum-mechanical analog to illustrate this point about signs. Consider the two forms of the quartic oscillator Hamiltonian:

$$H = \frac{1}{2}p^2 \pm \frac{1}{2}\omega^2 x^2 + \frac{\lambda}{4}x^4 \quad (1)$$

The dimensionless expansion parameter is  $\lambda/\omega^3$ , and can be large or small. For the positive sign, however, the qualitative behavior is not changed as this parameter goes from small to large; to be sure, perturbation theory does not converge (for any value), but it is Borel-summable, and any number of techniques (e.g., variational) serve to estimate the energy levels, etc, with acceptable accuracy. But for the negative sign the potential has two wells, Borel summation fails, and the system behavior is quite different for large and small values of  $\lambda/\omega^3$ . For large values there is a high barrier separating the two wells and these communicate with each other only by exponentially-suppressed tunnelling, but for small values the barrier height  $\lambda/\omega^2$  is less than the perturbative energy scale  $\omega$ , and the hump separating the two wells is only a small perturbation. In YMCS gauge theory the expansion parameter is  $Ng^2/m$  (more generally,  $Ng^2/p$  at momentum scale  $p$ ), or equivalently  $N/k$ , and the negative sign in the quartic oscillator potential (1) corresponds to asymptotic freedom. By analogy, one expects different phases as  $k$  varies relative to  $N$ , as we shall argue for on other grounds. Correspondingly, if the dynamics of YMCS were not asymptotically-free, one might expect quite different behavior, with no essential difference between large and small  $k$ . (Ref. [18] has some remarks on what  $d = 3$  YM theory would look like in this case).

We now describe the three approaches taken in this paper:

1. **Gauge-Invariant One-Loop Self-Energy.** For details see Section 2. One can define from the  $S$ -matrix for any process propagators and vertices whose proper parts are completely independent of the chosen gauge, and which satisfy ghost-free Ward identities[5, 20]. These were previously calculated[5, 20, 6, 18] at one-loop order for  $d = 3, 4$  YM theory. In the present work we calculate the gauge-invariant proper self-energy for YMCS theory, extending the work of Pisarski and Rao[2], which is a calculation of the conventional self-energy in the Landau gauge. Our extension is a straightforward if lengthy piece of work. We quote here some useful results. For YM theory with no CS term, the inverse propagator with gauge-invariant self-energy,  $\hat{\Delta}_{ij}^{-1}(p)$ , is:

$$\hat{\Delta}_{ij}^{-1} = (p^2\delta_{ij} - p_i p_j)d(p) + p_i p_j/\xi \quad (2)$$

$$d(p) = 1 - \pi b g^2/p \quad (3)$$

where the *gauge-invariant* constant  $b$  is:

$$b = 15N/32\pi \quad (4)$$

and  $p$  is the magnitude of the Euclidean three-momentum;  $\xi$  is an arbitrary gauge parameter. There is a tachyonic zero at  $p = \pi b g^2$ , because of the minus sign in (3); this zero leads to unphysical effects such as an imaginary running charge, as we describe later.

For YMCS theory at level  $k$ , there is a tree-level mass  $m$ :

$$m = k g^2/4\pi \quad (5)$$

and the propagator  $\hat{\Delta}_{ij}^{-1}$  has two terms, an even term with the kinematics of equation (2) and an odd term, to be given later. The even term we write as, with the notation of (2),

$$d(p) = 1 - \hat{A}(p), \quad (6)$$

with  $\hat{A}$  a complicated function given in Section 2. At  $p = 0$  this function has the value

$$\hat{A}(0) = 29N g^2/48\pi m = 29N/12k \quad (7)$$

and  $d(0)$  is negative (tachyonic) if

$$k \leq k_c = 29N/12. \quad (8)$$

Since  $\hat{A}(p)$  is a positive monotone decreasing function of  $p$  (behaving like  $1/p$  at large  $p$ ), there will always be a tachyonic zero in  $d(p)$  for some  $p$  if  $k \leq k_c$  as given in (8).

One may anticipate that this tachyonic zero in  $d$  will be removed[5, 6, 18] by the same mechanism in YMCS that operates for just YM theory with no CS term: A dynamical mass  $M$  is generated by condensate formation, in the case of YM theory replacing the  $1/p$  in  $d$  of equation (3) by something like  $1/(p^2 + 4M^2)^{1/2}$ , with  $M$  large enough to keep  $d$  positive at zero (and thus at any real Euclidean) momentum. Similarly,  $m$  in equation (7) should be replaced by something like  $(m^2 + M^2)^{1/2}$  with  $M$  large enough to keep  $d$  positive. We now discuss how this might happen.

2. **Mass and Condensate Generation At Small  $k$ .** These results are detailed in Section 3; we believe they are qualitatively correct although perhaps far from quantitative. Define the partition function  $Z$  and its logarithm as usual:

$$Z(k) = \int (dA) \exp(-S_{YMCS}) \quad (9)$$

$$Z(k) = \exp(-\int d^3x \epsilon) \quad (10)$$

It is easy to show that  $\epsilon$  is real and that  $\epsilon(k \neq 0)$  is always larger than  $\epsilon(k = 0)$ ; for small  $k$ , the fractional increase in  $\epsilon$  is  $O(k^2/N^2)$ .

This increase in  $\epsilon$  is in the direction to disfavor condensate formation, since condensate entropy tends to lower  $\epsilon$ . To understand this effect of non-zero  $k$  it is first necessary to review earlier work on condensate formation at  $k = 0$ .

Previously[21] we have given the exact form of the action for pure YM ( $k = 0$ ) theory in its dependence on the zero-momentum matrix elements of the condensate operator

$$\theta = \frac{1}{4}(G_{ij}^a)^2 \quad (11)$$

and shown that  $\epsilon$  (or equivalently the spatial density of  $\beta F$ , where  $F$  is the free energy, in thermal field theory) has a minimum for positive  $\langle\theta\rangle$ , and the minimum value is negative. This shows that a condensate has formed, with condensate entropy outweighing the positive internal energy. We have also given arguments [18, 21] consistent with Lavelle's[14] work that condensate formation drives the generation of a dynamical mass  $M$ , with  $M$  depending on  $\theta$  as  $(g^2\theta)^{1/4}$ , as naive dimensional reasoning predicts. A simple crude model of the free energy  $\epsilon$  was constructed[18, 21], based on adding the mass  $M$  by hand to the gauge-invariant propagator  $\hat{\Delta}$  discussed above, and the resulting form for the free energy was consistent (in a non-trivial way!) with the required[21] dependence on  $\theta$ . The addition of mass to this propagator was justified by an earlier[6] investigation of a non-linear gauge-invariant Schwinger-Dyson equation for  $\hat{\Delta}$ , which showed that, because of the sign of the  $b$  term in equation (3), this equation required a dynamical mass. The value of the mass could not be predicted, but could be bounded below; the bound is roughly  $2bg^2$  in terms of  $b$  of equation (4).

In the present work we extend these considerations to YMCS, in a crude way. To be more exact would require once again consideration of the non-linear Schwinger-Dyson equation for the gauge-invariant propagator, which we have not yet attempted. The same general effects are operative; because of the sign structure of the one-loop perturbative propagator, would-be tachyonic effects require more than just the CS mass  $m$  for their cure when  $k$  is small. It is plausible that the outcome of the Schwinger-Dyson equation, if really solved, would be to replace the perturbative YMCS mass  $m$  by a nonperturbative value of  $O((m^2 + M^2)^{1/2})$ , where now  $M$  stand for the contribution of the condensate to the mass; that is,  $M \sim (g^2\theta)^{1/4}$  as before. Now we simply take the one-loop YMCS propagator with this replacement for the mass, and repeat the earlier[18, 21] work on pure YM theory. The result is consistent with the general remarks made above, in that the lowest-order correction is  $O(k^2/N^2)$  and increases the free energy. As  $k$  increases a critical value  $k_c$  is reached where the free energy is positive and the condensate and its associated dynamical mass vanish. Because our construction of the free energy is based on a one-dressed-loop ap-

proximation we find the same value of  $k_c$  as in one-loop perturbation theory (see (8)). We also find a critical exponent for dynamical mass generation:

$$M \sim (k_c - k)^{1/2} \text{ for } k < k_c \quad (12)$$

3. **Quantum and Classical Solitons.** More evidence for the two-phase structure of YMCS theory is found by looking for classical and quantum solitons. In the classical theory, with no dynamical mass  $M$ , this has already been done by D'Hoker and Vinet[17]. The results are that there are no Euclidean classical solitons with finite action, either of the Abelian vortex type or of the sphaleron (spherically-symmetric) type. In the latter case, a peculiar solution exists in which the soliton field has an accumulation point of zeroes at the origin; its action is infinite. The absence of classical solitons with finite action is consistent with the idea that the  $k > k_c$  phase has no condensate and can be treated semiclassically.

In the other phase, with a dynamical mass  $M$ , these solitons are profoundly modified, and one can find quantum solitons of both the vortex type and the sphaleron type.<sup>1</sup> These are found as classical solutions of an effective action, containing not only the usual YM and CS terms but also [22] a gauge-invariant mass term for the dynamical mass; this is just a gauged non-linear sigma model. It is well-known that this added mass term does not lead to a perturbatively-renormalizable theory, because of divergences associated with the implicit assumption that the mass  $M$  is a “hard” mass, surviving at large momentum. In fact, the dynamical mass is a soft mass, vanishing[23, 14] like  $p^{-2}$  (modulo logarithms) at large momentum  $p$ . Since we should not and will not use the effective action beyond the classical level, we will treat the mass  $M$  as a constant, although this introduces a spurious logarithmic divergence in the action associated with the gauged non-linear sigma model. The true action, with all quantum corrections, is finite.

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<sup>1</sup>The author and B. Yan have made a preliminary study of the analogous solitons in Euclidean YMCS theory with a *real* Chern-Simons coefficient. The CS coefficient then serves as a Lagrange multiplier for specifying the expectation value of the CS action. The resulting solitons are real, and the vortex is twisted; this twist contributes to the CS term[8, 9, 11]. Results will be reported elsewhere.



With the dynamical mass added, there are actually two different propagator-pole masses, both non-zero; as  $M$  approaches zero, the heavier mass becomes the CS mass  $m$  and the lighter one approaches zero (which it will be recalled is also a propagator mass of classical YMCS theory). At  $m = 0$ , both masses merge into the dynamical mass  $M$ . In addition to these poles, there is also a zero-mass excitation corresponding roughly to a Goldstone boson (although there is no symmetry breaking), but this is better identified as a long-range pure-gauge excitation of the gauge potential which yields such effects as [7, 9] confinement and a string tension.

The effective action is complex because of the CS term, which has an  $i$  factor in Euclidean space. As a result, the general soliton solution of the effective action is also complex. In view of the fact that the partition function is real, there can be solitons which have complex action but which occur in complex-conjugate pairs (equivalently paired under  $k \rightarrow -k$ ), or complex solitons which have real action. We have only found the latter, both for the vortex and for the sphaleron. These two solitons have different fates as the dynamical mass  $M$  goes to zero, that is, as  $k \rightarrow k_c$ . The vortex is Abelian, and is an extension of the well-known Nielsen-Olesen vortex; its gauge potentials can be explicitly written in terms of Hankel functions of imaginary argument. This vortex involves both of the pole masses mentioned above in such a way that it has finite YMCS action by virtue of a cancellation between terms involving these different masses, and smoothly vanishes as  $M$  goes to zero. The sphaleron soliton is rather different; it can only be found numerically. We have found numerical solutions for small values of  $m/M$ , which will be displayed in Section 4. As this parameter increases it becomes increasingly harder to find solutions to the equations of motion. We have used a simple variational approach for larger values of  $m/M$  which suggests that the sphaleron soliton becomes singular at  $m/M \gtrsim 0.5$ ; the singularity is of the same general type as found by D'Hoker and Vinet[17].

In both cases, the solitons we have found decouple from the theory when  $m/M$  or equivalently  $k/N$  is large enough, which is consistent with the idea that YMCS theory becomes semiclassical at large  $k$ .

To make further progress would require an accurate evaluation of the

contribution of the solitons, including their entropy, to the partition function. This is difficult and uncertain, and we have not attempted it.

It appears that YMCS theory, while not possessing any immediately obvious applications to particle physics, is an interesting testing ground for various non-perturbative phenomena of field theory. One check of the ideas presented here might be through lattice-gauge simulations. Other directions worthy of investigation are to search for an approximate duality between  $k$  and  $N$ , in the spirit of Seiberg-Witten[24] duality, or to extend the theory to supersymmetric YMCS, in light of the fact that chiral fermions can induce a CS term[25].

## 2 Gauge-Invariant One-Loop Perturbation Theory

In this Section we will calculate a *gauge-invariant* one-loop self-energy for YMCS theory, using the pinch technique[5, 20]. The pinch technique adds to the conventional self-energy some new terms defined by the  $S$ -matrix; these new terms, among other things, cancel the dependence of the conventional self-energy on the choice of gauge<sup>2</sup>. Since the usual self-energy has already been calculated by Pisarski and Rao[2] in the Landau gauge, we need only compute the extra terms, also in the Landau gauge. The result is independent of the gauge, as one may readily check by adding the appropriate gauge terms to both the Pisarski-Rao terms and the terms we find here.

To understand the pinch technique, consider the one-loop graphs for the  $S$ -matrix element of two-particle scattering, where the external lines can have different masses and spins, be in arbitrary representations of  $SU(N)$ , etc. The only requirement is that they be on-shell. All these graphs for fermions (except for external-line wave-function renormalization) are shown in Fig. 1a, b, c, d, f, g, i. (For the moment ignore the graphs with heavy vertices.) The conventional propagator comes from Fig. 1a, b, c with, of course, no external lines attached. But just the sum of these three graphs is

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<sup>2</sup>It has been remarked[18] that at least at one-loop level the pinch technique gives the same result as the Feynman-gauge background field technique.

not gauge-invariant; all the rest of the graphs must be added to get a gauge-invariant  $S$ -matrix. The pinch technique identifies parts of graphs d, f, g, and i which act exactly like propagator parts and which, when added to the usual terms, yield a gauge-invariant result. This must happen because all terms in the  $S$ -matrix with the kinematic structure of propagator exchange between bare vertices have different dependence on the kinematic variables than any other set of terms (e. g., they are independent of external-line masses and of energy variables, except for trivial external-line wave functions).

Consider now the vertex labeled **i** in Fig. 1d, associated with a factor  $\gamma_i$ . There will be a term  $\sim k_i$  multiplying this vertex coming from gauge-boson propagator parts or from three-boson vertices. This triggers the Ward identity

$$k_i \gamma_i = S^{-1}(p) - S^{-1}(p - k) \quad (13)$$

where  $S(p)$  is the external-line propagator of momentum  $p$ . But  $p$  is on-shell, so  $S^{-1}(p) = 0$ , and the other term in (13) cancels out the propagator of momentum  $p - k$ . The result is a graph with the structure shown in Fig. 1e, where the heavy vertex indicates a pinch has taken place. Similarly, in any but the Feynman gauge the box graphs Fig. 1f, g and the vertex graph Fig. 1i have pinch parts, as shown in Fig. 1h, j.

We will next report on the calculation of the pinch graphs (Fig. 1d, h, j), which is straightforward but somewhat lengthy. One comment is needed about the pinch in graph Fig. 1i. The group-theoretic factor of this vertex is  $C_R - N/2$ , where  $C_R$  is the Casimir invariant for the external lines in representation  $R$  of  $SU(N)$ , and the  $N/2$  is half the adjoint Casimir. The  $C_R$  part cancels the gauge dependence in external-line wave-function graphs (not shown), which occur with weight  $1/2$  twice for each line. The only pinch cancellation relevant to the propagator comes from the  $N/2$  part, and therefore we define the pinch graph Fig. 1j to have the group-theoretic factor  $-N/2$ . Note that this graph vanishes, by dimensional regularization, in the  $m = 0$  (pure YM) theory, but not in YMCS theory. Also note that graphs Fig. 1e, j must be multiplied by two because of the two external lines.

First we establish some notation. Define the scalar one-loop integral with two masses:

$$J(m_1, m_2; p) = \frac{1}{(2\pi)^3} \int d^3q \frac{1}{(q^2 + m_1^2)((q + p)^2 + m_2^2)} \quad (14)$$

so that  $J(0, 0; p) = 1/8p$  and

$$J(0, m; p) \equiv J_1 = \frac{1}{16p} + \frac{1}{8\pi p} \arctan\left(\frac{p^2 - m^2}{2pm}\right) \quad (15)$$

$$J(m, m; p) \equiv J_2 = \frac{1}{4\pi p} \arctan\left(\frac{p}{2m}\right) \quad (16)$$

We use the Pisarski-Rao[2] bare propagator and vertex:

$$\Delta_0(p)_{ij} = (\delta_{ij} - p_i p_j / p^2 - m \epsilon_{ija} p_a / p^2) \frac{1}{p^2 + m^2} + \xi p_i p_j / p^4 \quad (17)$$

$$\Gamma_{ijk}(p, q, -p - q) = \delta_{jk}(2q + p)_i - \delta_{ik}(2p + q)_j + \delta_{ij}(p - q)_k + m \epsilon_{ijk} \quad (18)$$

(The ghost propagator and vertex and the four-point vertex are the same as in pure YM theory). In (17),  $\xi$  is a gauge parameter.

The gauge-invariant propagator inverse is:

$$\hat{\Delta}^{-1}(p)_{ij} = \Delta_0^{-1}(p)_{ij} - \hat{\Pi}(p)_{ij} \quad (19)$$

where the bare inverse is

$$\hat{\Delta}_0^{-1}(p)_{ij} = (p^2 \delta_{ij} - p_i p_j) + m \epsilon_{ija} p_a + p_i p_j / \xi \quad (20)$$

and the self-energy has the conserved<sup>3</sup> form:

$$\hat{\Pi}(p)_{ij} = (p^2 \delta_{ij} - p_i p_j) \hat{A}(p) + m \epsilon_{ija} p_a \hat{B}(p). \quad (21)$$

The propagator itself is:

$$\begin{aligned} \hat{\Delta}(p)_{ij} = & (\delta_{ij} - p_i p_j / p^2) \frac{1}{(1 - \hat{A})(p^2 + m_R^2)} \\ & - m_R \epsilon_{ija} p_a \frac{1}{p^2 (1 - \hat{A})(p^2 + m_R^2)} + \xi p_i p_j / p^4 \end{aligned} \quad (22)$$

and the renormalized running mass  $m_R$  is:

$$m_R(p) = m \left( \frac{1 - \hat{B}}{1 - \hat{A}} \right) \quad (23)$$

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<sup>3</sup>Longitudinal terms cannot contribute to the  $S$ -matrix of Fig. 1.

Just as one does for QED, one can define running charges (in this case, two of them, one for the parity-even exchange and one for the parity-odd) via

$$g^2 \hat{\Delta}(p)_{ij} = \delta_{ij} \frac{g_R^2(p)}{p^2 + m_R^2} + \dots \quad (24)$$

(We do not write the parity-odd term explicitly.) Clearly,

$$g_R^2 = \frac{g^2}{1 - \hat{A}} \quad (25)$$

What we find in one-loop perturbation theory is that for  $k < 29N/12$ , there is a real value of the momentum  $p$  for which  $1 - \hat{A}$  vanishes, and this quantity is negative for smaller  $p$ . Evidently, this leads to tachyonic poles in both the running charge and in the running mass (but not necessarily in the propagator itself unless  $1 - \hat{B}$  vanishes at the same momentum); for momenta smaller than the pole momentum, the running charge is imaginary. Such behavior is physically unacceptable, and calls for dynamical mass generation as discussed in Section 3.

Here are the one-loop results, beginning with the Pisarski-Rao calculation. Letters in parentheses refer to the appropriate graphs of Fig. 1.

$$\begin{aligned} \hat{A}(a, b, c) = & \frac{Ng^2}{32\pi m} \left\{ 5 + 11m^2/p^2 - \frac{\pi}{2m^3 p^3} [m^2(2p^4 + 13p^2 m^2/2 \right. \\ & + 7m^4/2) - (p^2 - 7m^2)(p^2 + m^2)^2(-1/2 + 8pJ_1) \\ & \left. - 4p(p^4 - 13p^2 m^2 + 4m^4)(p^2 + 4m^2)J_2] \right\} \end{aligned} \quad (26)$$

$$\begin{aligned} \hat{B}(a, b, c) = & -\frac{Ng^2}{16\pi m} \left\{ 2 + m^2/p^2 + \frac{\pi}{4m^3 p^3} [m^2(p^4 + p^2 m^2 - m^4) \right. \\ & + (3p^2 - m^2)(p^2 + m^2)^2(16pJ_1 - 1) \\ & \left. - 24p^3(p^2 - 2m^2)(p^2 + 4m^2)J_2] \right\} \end{aligned} \quad (27)$$

Next, the pinch contributions, listed separately for graph e and for graphs h + j:

$$\begin{aligned} \hat{A}(e) = & Ng^2 \{ 9m/16\pi p^2 + p/8m^2 - 7/16\pi m + 3J_2 \\ & + J_1[4p^2(m^2 - p^2) - (p^2 + m^2)(p^2 + 5m^2)]/4p^2 m^2 - 2(p^2 - m^2)Q \} \end{aligned} \quad (28)$$

where

$$Q = -1/32\pi m p^2 + (p^2 + m^2)^2 J_1/8p^2 m^4 - (p^2 + 4m^2)J_2/16m^4 - p/128m^4 \quad (29)$$

$$\begin{aligned} \hat{B}(e) = & Ng^2\{-p^3/8m^4 - 15/16\pi m + m/16\pi p^2 \\ & + J_1(7/2 + 15p^2/4m^2 - m^2/4p^2 + 2p^4/m^4) \\ & + J_2(3 - 15p^2/4m^2 + p^4/m^4) - 4p^2Q \end{aligned} \quad (30)$$

$$\hat{A}(h+j) = Ng^2\{1/4\pi m - (p^2 - m^2)/m^2(J_1 - J_2) + (p^2 - m^2)Q\} \quad (31)$$

$$\begin{aligned} \hat{B}(h+j) = & Ng^2\{1/4\pi m + p(p^2 - m^2)/16m^4 \\ & + p^2(p^2 + m^2)(J_2 - J_1)/2m^4 + 2p^2Q \end{aligned} \quad (32)$$

The functions  $\hat{A}, m\hat{B}$  have no infrared singularities at  $m = 0$  for finite  $p$ , or at  $p = 0$  for finite  $m$ . So a perturbation series in  $m$  or  $k$  can in principle be written as long as  $p$  is large enough<sup>4</sup>. We begin by looking at the small  $m$ , or equivalently large  $p$  case, which amounts to a perturbation expansion jointly in  $Ng^2/p$  and in  $m/p$ . We will save only the lowest-order terms in these parameters. Let us add to the bare inverse propagator (20) the sum of parity-even self-energies in (26), (28), and (31) and parity-odd self-energies in (27), (29), and (32) at  $m = 0$  to find:

$$\hat{\Delta}^{-1}(p)_{ij} = (p^2\delta_{ij} - p_i p_j)(1 - 15Ng^2/32p) + \epsilon_{ija} p_a g^2\left(\frac{k+N}{4\pi}\right) \quad (33)$$

One recognizes here, as expected, the pure YM self-energy already given in (3) as the coefficient of  $\delta_{ij}$ . In the  $\epsilon$  term we have rewritten  $m$  of (20) as  $kg^2/4\pi$ , and we observe that there is a  $m^{-1}$  contribution to  $\hat{B}$  at small  $m$  which cancels out the kinematic  $m$  factor in (20), giving rise to the  $N$  term in (33). The result is that  $k$  is renormalized at one-loop level to  $k + N$ , a well-known[2] result. This is, in fact, the exact renormalization of  $k$  to all orders of perturbation theory[2]. That the renormalization of  $k$  is solely a

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<sup>4</sup>Pisarski and Rao[2] have argued that only in the Landau gauge is the conventional propagator, Fig. 1a, b, c, infrared-finite at  $p = 0$ , although the  $S$ -matrix has only massive singularities. Thus we expect our  $S$ -matrix-derived propagator to have only massive singularities.

mass renormalization can be traced to the QED-like Ward identity of the type  $Z_1 = Z_2$  which holds for the gauge-invariant Green's functions of the pinch technique.

Now we will look at small momentum for finite mass. It is straightforward to check that  $\hat{A}(p)$  is positive and monotone decreasing (it vanishes at rate  $1/p$  at large  $p$ ), so the possibility of tachyonic behavior can be examined by looking at  $p = 0$ . At this point there is a perturbative expansion in powers of  $Ng^2/m$ . The key result comes from the parity-even self-energy at zero momentum:

$$1 - \hat{A}(p = 0) = 1 - 29N/12k. \quad (34)$$

So if

$$k \leq k_c \equiv 29N/12, \quad (35)$$

there is a tachyonic zero in the parity-even self-energy, or as pointed out in equations (23), (25), a pole in the running mass and charge (unless  $\hat{B}$  were also to have a zero at the same  $p$ ).

We conclude that one-loop perturbation theory fails for  $k \leq k_c$ . Just because it does fail, we do not have a reliable value for the particular value of  $k$  where perturbation theory goes wrong. For example, one might argue that  $k$  is renormalized to  $k + N$  so that perhaps the critical value of  $k$  is found from  $k + N \leq 29N/12$ , or  $k \leq 17N/12$ . This suggests that we might be within a factor of two of understanding just what  $k_c$  is; to do better would require a fully non-perturbative treatment. Although we are not in a position to give this, we will in the next Section give some qualitative arguments about what happens non-perturbatively when  $k$  is near  $k_c$ .

### 3 Non-Perturbative Behavior Near the Critical Value of $k$

Our objective here is to give a qualitative description of how YMCS theory behaves as  $k$  goes from small to large. We will build on previous works [18, 21] which have given a similar description for pure YM theory (that is,  $k = 0$ ). It was argued in these works that there is a condensate of the operator  $\theta = (G_{ij}^a)^2/4$  with positive vacuum expectation value, corresponding to a negative value of the vacuum energy  $\epsilon$ , with  $\epsilon = -\langle\theta\rangle/3$ . The basis for

these arguments is an exact form of the effective action as it depends on zero-momentum matrix elements of  $\theta$ . The condensate is self-consistently related to the generation of a dynamical gauge-boson mass  $M$ , with  $M \sim (g^2 \langle \theta \rangle)^{1/4}$ . The negative value of  $\epsilon$  is associated with configurational entropy of the condensate. We first review these  $k = 0$  considerations, beginning with the definition of the gauge potentials and action for general  $k$  via:

$$\vec{A}(\vec{x}) = \frac{g\lambda_a}{2i} \vec{A}^a(\vec{x}) \quad (36)$$

$$G_{ij}(\vec{x}) = \partial_i A_j(\vec{x}) - \partial_j A_i(\vec{x}) + [A_i(\vec{x}), A_j(\vec{x})] \quad (37)$$

where the  $\lambda_a$  are the usual generators of the group in the fundamental representation. The action is:

$$S_{YMCS} = - \int d^3x \frac{1}{2g^2} \text{Tr} G_{ij}^2 + 2\pi i k W \equiv S_{YM} + 2\pi i k W \quad (38)$$

and the Chern-Simons term is

$$W = -\frac{1}{8\pi^2} \int d^3x \epsilon_{ijk} \text{Tr} (A_i \partial_j A_k - \frac{2}{3} A_i A_j A_k). \quad (39)$$

In these equations and in what follows a potential or field with a group index, such as  $\vec{A}^a$ , is canonical and Hermitean, with  $\vec{A}^a$  having mass dimension 1/2; matrix-valued potentials and fields are multiplied by  $g$  and are anti-Hermitean. Note that  $\theta$  is the square of the canonical field strength and that  $S_{YM} = \int d^3x \theta$ . The partition function and vacuum energy density are defined as in (9) and (10), repeated here for convenience:

$$Z = \int (dA) \exp(-S_{YMCS}) = \exp(- \int d^3x \epsilon) \quad (40)$$

At  $k = 0$ , the effective action<sup>5</sup>  $\Gamma(\theta)$  as it depends on the zero-momentum matrix elements of  $\theta$  is easily found[21] by introducing a constant source  $J$  into  $Z$  and Legendre transforming:

$$Z(J) \equiv \int (dA) \exp\left\{ \int d^3x \frac{1}{2g^2} \text{Tr} G_{ij}^2 (1 - J) \right\}; \quad (41)$$

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<sup>5</sup>The vacuum energy  $\int d^3x \epsilon$  is the effective action evaluated at its minimum in  $\theta$ .



that is,  $S_{YM} = \int \theta$  so adding the constant source is just the same as the change  $g^2 \rightarrow g^2(1 - J)^{-1}$ . On dimensional grounds,  $\epsilon \sim g^6$  and thus

$$\epsilon(J) = \epsilon(J = 0)(1 - J)^{-3} = -\frac{\langle \theta \rangle}{3}(1 - J)^{-3} \quad (42)$$

(by  $\langle \theta \rangle$  we mean the expectation value of  $\theta$  for  $J = 0$ ; see (43, 46) below). Since  $\theta = -\partial\epsilon/\partial J$ , one has

$$\theta = \langle \theta \rangle (1 - J)^{-4}. \quad (43)$$

Legendre transform, as usual:

$$\begin{aligned} \Gamma(\theta) &= \epsilon(J) + \int d^3x J\theta \\ \partial\Gamma/\partial\theta &= \int d^3x J. \end{aligned} \quad (44)$$

One finds from (42-44) that

$$\Gamma(\theta) = \int d^3x \left( \theta - \frac{4}{3} \theta^{3/4} \langle \theta \rangle^{1/4} \right), \quad (45)$$

with a minimum ( $J = 0$ ) value

$$\Gamma(\langle \theta \rangle) = -\frac{1}{3} \langle \theta \rangle \quad (46)$$

We have given earlier[18, 21] simple models which realize this structure for  $\Gamma(\theta)$ , based on a one-dressed-loop CJT[26] potential where the dressed propagator has a dynamical mass  $M$  depending on  $\theta$ . The dressed propagator in question is the gauge-invariant one called  $\hat{\Delta}$  above. Lavelle has shown[14] that this propagator has a gauge-invariant contribution<sup>6</sup> from  $\theta$  of the form

$$\hat{\Delta}_{ij}(p) = (\delta_{ij} - p_i p_j / p^2) d^{-1}(p) \quad (47)$$

$$d(p) = p^2 + a g^2 \theta / p^2 \quad (48)$$

at large momentum  $p$ , where

$$a = \left( \frac{58}{15} \right) \frac{N}{N^2 - 1}. \quad (49)$$

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<sup>6</sup>Non-gauge-invariant condensates, such as ghosts, cancel out in  $\hat{\Delta}$  but not in the usual propagator.

The second term expresses the large-momentum behavior of the (squared) dynamical mass  $M$ , just as a constituent quark mass is related to its corresponding condensate. At small momentum the  $1/p^2$  behavior is modulated by the dynamical mass  $M$  (roughly,  $p^2 \rightarrow p^2 + M^2$ ), as could be described in principle by a non-linear Schwinger-Dyson equation (see Ref. [5, 6] for an attempt in this direction which maintains gauge invariance). Unfortunately, we do not really know how to do this, and we will instead make the crude approximation of evaluating the masslike term in (48) at  $p^2 = M^2$  to find  $M$ , or in other words,

$$M = (ag^2\theta)^{1/4}. \quad (50)$$

A more accurate treatment would supply a constant factor of  $O(1)$  in this relation between mass and condensate.

The one-dressed-loop CJT approximation to the vacuum energy  $\epsilon$  is really just a one-loop background field calculation[27] with the condensate field as the background field, which amounts to attaching external fields to the one-loop pinch-technique proper self-energy. Instead of calculating the self-energy perturbatively as we did in Section 1, we give a mass  $M$  by hand to the lines in this loop (for details, see Ref. [18, 21]), which gets rid of the perturbative infrared divergences and mimics the result of solving the Schwinger-Dyson equations[5, 6]. The result is

$$\int d^3x \epsilon = \frac{V}{(2\pi)^3} \int d^3p \frac{1}{4} \tilde{G}_{ij}^a(p) (1 - \hat{A}(p)) \tilde{G}_{ij}^a(-p) \quad (51)$$

where  $V = \int d^3x$ ,  $\tilde{G}$  is the Fourier-transformed field, and  $\hat{A}(p)$  is the scalar integral for the pure YM theory pinch self-energy but with massive propagators:

$$\hat{A}(p) = \left(\frac{15}{4}\right) N g^2 J_2 \quad (52)$$

(recall that  $J_2$  is the  $d = 3$  scalar one-loop graph with two massive lines, as given in (14) and (16)). At  $M = 0$  this yields the massless pure-YM result previously given in (3), (4), while at  $p = 0$  one finds equation

$$1 - \hat{A}(0) = 1 - 15 N g^2 / 32 \pi M. \quad (53)$$

A simple qualitative approximation to the one-dressed-loop vacuum energy (51) comes from replacing  $\hat{A}(p)$  by  $\hat{A}(0)$  in that equation. One readily checks

that, with the mass  $M$  given by (50), the resulting one-dressed-loop expression (51) for  $\epsilon$  is of the desired form (45), with a specific and probably rather inaccurate value for  $\epsilon$  as a positive constant times  $(Ng^2)^3(N^2-1)$ . The crucial minus sign in (45) comes from the positive nature of  $\hat{A}$  in (53), corresponding to asymptotic freedom.

Now turn to YMCS theory, with  $k \neq 0$ . It is easy to show that finite  $k$  increases the vacuum energy, at a quadratic rate for small  $k$ . Note that for every (real) gauge potential  $\vec{A}^a(\vec{x})$  occurring in the path integral for  $Z$  there is another configuration  $-\vec{A}^a(-\vec{x})$  for which the Yang-Mills part of the action, and the measure, are unchanged, but for which the Chern-Simons term changes sign. This means that  $Z$  is even in  $k$ , and we can write

$$Z = \int (dA) \exp(-S_{YM}) \cos(2\pi kW) \quad (54)$$

and it is evident that  $Z(k \neq 0) < Z(k = 0)$ . A formal expansion for small  $k$  yields:

$$\int d^3x \epsilon(k) = \int d^3x \epsilon(k=0) + 2\pi^2 k^2 \langle W^2 \rangle \quad (55)$$

where the expectation value is taken in the  $k=0$  theory. It is easy to check that  $\langle W \rangle = O(k)$  for small  $k$ , so to lowest order

$$\langle W^2 \rangle = \langle (W - \langle W \rangle)^2 \rangle \equiv \int d^3x \chi_{CS} \quad (56)$$

that is, the correction term depends on the gauge-invariant Chern-Simons susceptibility  $\chi_{CS}$ , which has been studied in Ref. [21] for pure  $SU(2)$  YM theory. Standard  $N$ -counting arguments show that

$$\epsilon \sim (N^2 - 1)(Ng^2)^3, \quad \chi_{CS} \sim (N^2 - 1)Ng^6 \quad (57)$$

so that the fractional increment in  $\epsilon$  is  $O(k^2/N^2)$ . It is not hard to believe that there is some value of  $k \sim N$  for which  $\epsilon = 0$  and the condensate entropy no longer overcomes the free energy. But for smaller values of  $k$ ,  $\epsilon$  remains negative and there is a dominant condensate as well as a dynamical mass.

We give a very simplified model of how that dynamical mass might interact with the Chern-Simons mass. The idea, as before, is that the non-linear Schwinger-Dyson equations for the YMCS pinch propagator will be singular if  $m$  is too small, just as happens for the pure-YM theory, and the true mass

of the theory will be augmented by a dynamical mass. Actually to solve these Schwinger-Dyson equations is beyond our hopes at the moment, so we will proceed with what we hope are sensible assumptions about what really happens and fold in the dynamical mass  $M$  more or less by hand into the  $k = 0$  results given above. As described in Section 4 below, it is possible to add a gauge-invariant mass term[22, 5] to the YMCS action, yielding an effective action describing the quantum-mechanical generation of dynamical mass. The resulting effective propagator is easy to describe in terms of the self-energies<sup>7</sup>  $\hat{A}$ ,  $\hat{B}$  of Section 1 (equations (20-22)): Just take

$$1 - \hat{A}(p) = 1 + (M^2/p^2), \quad \hat{B} = 1. \quad (58)$$

One finds from (22) that the parity-even propagator now has two Minkowski-space non-tachyonic masses  $\mu_{\pm}$ :

$$\mu_{\pm} = \frac{1}{2}[\pm m + (m^2 + 4M^2)^{1/2}] \quad (59)$$

which are also the physical masses coming from the Higgs mechanism[2]. The mass  $\mu_-$  goes to zero at  $M = 0$ , and  $\mu_+$  goes to  $m$  in this limit. We will make the *Ansatz*, unjustified by any deep analysis, that a qualitative description of the behavior of the YMCS pinch propagator at small momentum is found by replacing  $m$  as it appears in the zero-momentum one-loop propagator (see (6), (7), (34)) by some mass which, like  $\mu_+$ , approaches  $m$  as  $M \rightarrow 0$ . (Recall that even though there is a zero-mass pole in the bare YMCS propagator, it does not appear in physical quantities like the  $S$ -matrix, so a mass like  $\mu_-$  is not a good candidate.) It is hardly justified to argue for the specific form of  $\mu_+$ , and instead we use the replacement

$$m \rightarrow \mu \equiv (m^2 + M^2)^{1/2} \quad (60)$$

which gives the correct small- $m$  or small- $k$  limit, and for large  $m$  yields a mass which receives quadratic corrections from  $M$ . This also holds for  $\mu_+$  of (59), but with a different numerical coefficient.

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<sup>7</sup>The form given for  $\hat{A}$ , of opposite sign and with one more power of  $p$  than the one-loop perturbative result in (33), is characteristic of what two- and higher-loop graphs contribute to mass generation. It is what we expect from the condensate term in (48) at small momentum.

We want to investigate the behavior of the vacuum energy for  $k \neq 0$  in the same spirit as before, in a one-dressed-loop approximation. To simplify matters we will only take explicit account of the parity-conserving part of the one-loop pinch self-energy, which one can check does not change the qualitative behavior we find. Our strategy, then, is to take the  $k = 0$  expression (51) for the effective action and in it to make the replacement of

$$1 - \hat{A}(p) \rightarrow 1 - \hat{A}(0) \approx 1 - 29Ng^2/48\pi\mu \quad (61)$$

where we used (7) for the dependence of  $\hat{A}(0)$  on  $m$ . Combine equations (50, 51, 60, 61) to find:

$$\epsilon = \theta \left\{ 1 - \frac{29Ng^2}{48\pi} [m^2 + (ag^2\theta)^{1/2}]^{-1/2} \right\}. \quad (62)$$

At  $m$  or  $k = 0$  this has the required pure-YM form (45), and for small  $k$  the fractional corrections to  $\epsilon$  are also as required, to increase it by  $O(k^2/N^2)$ .

Now let  $m$  increase, and observe that when  $k$  reaches the critical value given earlier in equation (35), namely  $29N/12$ , the vacuum energy  $\epsilon$  is positive for any positive value of  $\theta$ , and has its minimum at  $\theta = 0$ . The condensate and dynamical mass are now gone, and semiclassical methods should serve to calculate all properties of YMCS theory. But for smaller  $k$ ,  $\epsilon$  has a minimum at some positive value of  $\theta$  and at the minimum  $\epsilon$  is negative. To find that minimum, introduce the variables

$$x = (ag^2\theta)^{1/2}/m^2, \quad \alpha = 29N/12k. \quad (63)$$

The equation determining the value of  $x$  for which  $\epsilon$  is minimum is:

$$(1+x)^3 = \alpha^2 \left(1 + \frac{3x}{4}\right)^2. \quad (64)$$

At the critical value of  $k$ , which is  $\alpha = 1$ , the only real solution is  $x = 0$ . For  $k$  just below the critical value, or when  $\alpha - 1 \ll 1$ , one finds  $x \approx (4/3)(\alpha - 1)$  which corresponds to the critical behavior

$$M(\theta) \sim (\langle\theta\rangle)^{1/4} \sim (k_c - k)^{1/2}. \quad (65)$$

Finally, when  $k \ll k_c$ , the solution merges smoothly onto the  $k = 0$  model discussed above.

We certainly cannot expect the crude approximations of this section to be anywhere near quantitatively accurate, but we can hope that certain features are realistic. These might include the critical exponent  $1/2$  for the dynamical mass given in (65). It is unlikely that any real improvement in the situation will be gained by analytic techniques, so one might well look forward to lattice-gauge simulations of YMCS theory.

## 4 Quantum Solitons

D'Hoker and Vinet[17] have looked for Euclidean solitons of *classical* YMCS theory. It turns out that they found none with finite action, as we will review, either of the vortex type or of the sphaleron (i.e., spherically symmetric) type. Here we will investigate *quantum* solitons, that is, solitons arising as solutions of an effective action, to be treated classically, but containing an extra term summarizing the quantum-mechanical generation of a dynamical mass  $M$ . This effective action[5, 22, 13, 18] is:

$$S_{eff} = S_{YMCS} - M^2/g^2 \int d^3x Tr(U^{-1}D_i U)^2 \quad (66)$$

where  $D_i$  is the covariant derivative  $\partial_i + A_i$  and  $U$  is an  $N \times N$  matrix with the transformation law

$$U \rightarrow VU \quad (67)$$

under local gauge transformations

$$A_i \rightarrow VA_iV^{-1} + V\partial_iV^{-1} \quad (68)$$

The added mass term is just a gauged non-linear sigma model, and is locally gauge-invariant under (67, 68). The equations of motion are:

$$[D_i, G_{ij}] = m^2(D_j U)U^{-1} \quad (69)$$

$$[D_i, (D_i U)U^{-1}] = 0 \quad (70)$$

Note that the equation of motion (70) for  $U$  is compatible with the identity

$$[D_i, [D_j, G_{ij}]] \equiv 0 \quad (71)$$

As a result, not all the equations of motion are independent; those for  $U$  follow from those for  $A$ . We have already mentioned that the dynamical

mass  $M$  is not really a constant, as we will treat it here, but vanishes rapidly at large momentum (see (48)); in view of this rapid vanishing, the mass term of the effective action (66) is not really correct at short distance, and gives rise to spurious logarithmic divergences in the action for the solitons we find below. The true effective action is indeed finite, and we will refer to our solitons as having finite action because the YMCS part of the action is finite.

Let us begin by finding vortex solutions with Abelian holonomy which, by a local gauge transformation, can be chosen to be Abelian everywhere. We give only solutions for closed vortices and not for open vortices which terminate on monopoles. These solutions are:

$$A_i(x) = \frac{2\pi Q}{\mu} \oint dz_k \{ \epsilon_{ijk} \partial_j [\mu_- (\Delta_+(x-z) - \Delta_0(x-z)) + (+ \leftrightarrow -)] + i\delta_{ik} \mu_+ \mu_- [\Delta_+(x-z) - \Delta_-(x-z)] \} \quad (72)$$

In this equation, the integral is over a closed path, and  $Q$  is an anti-Hermitean generator of the gauge group such that  $\exp(2\pi Q)$  is in the center of the group<sup>8</sup>. The quantities  $\mu_{\pm}$  are the masses given in equation (59), and  $\mu = \mu_+ + \mu_-$ . The propagators  $\Delta_{\pm}$ ,  $\Delta_0$  are the usual Euclidean  $d = 3$  free propagators of mass  $\mu_{\pm}$  and 0, and the massless propagator gives the contribution of the  $U$  field. This, of course, is the only surviving term at large distances from the vortex closed path, and it is a singular pure gauge term. The specific combination of propagators used is chosen so that the vortex potential gives rise to finite YMCS action (per unit length). Note that, in the  $\epsilon_{ijk}$  term of (72) the combinations  $\Delta_{\pm} - \Delta_0$  and  $\Delta_+ - \Delta_-$  are finite at short distances.

In the limit of pure YM theory ( $\mu_+ = \mu_- = M$ ), the vortex solution reduces to the previously-given [7, 8, 9] vortex which has only the  $\epsilon_{ijk}$  term in (72). In the opposite limit  $M = 0$ ,  $\mu_+ = m$ ,  $\mu_- = 0$ , the vortex smoothly disappears. It is apparent that the  $\delta_{ik}$  term and the  $\Delta_+$  part of the  $\epsilon_{ijk}$  term disappear, because of the  $\mu_-$  factor (the propagators are not that singular in the limit). The remaining  $\epsilon_{ijk}$  term proportional to  $\mu_+$  disappears because  $\Delta_- \rightarrow \Delta_0$  in the limit. This is not to say that one cannot find any vortex solution when  $M = 0$ ; in fact, one can, but it involves only one mass  $m$ , not two, and the cancellations which occur in (72) to yield finite action cannot be expressed. The result is a vortex with a singular field strength coming from the  $\delta_{ik}$  term.

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<sup>8</sup>This normalization is needed so that a gauge potential transported around a closed path linking the vortex will be single-valued.

As long as  $M \neq 0$ , there is a long-range pure-gauge part of the vortex (72) which can, in principle, contribute to confinement by a topological linking of vortices with a Wilson loop[7, 9]. But this can only happen if there is in fact a condensate of vortices, and it is the non-zero condensate which is responsible for the existence of  $M$ . So we expect a string tension as long as there is a dynamical mass. Based on earlier[9] work on the string tension in pure-YM theory, we might expect this quantity to scale like  $M^2$ , or in view of (65), like  $k_c - k$  near the transition point.

Note that the vortex is complex, with the real part even and the imaginary part odd under  $k \leftrightarrow -k$ . The action, including the Chern- Simons term, is real, however.

Next we turn to spherically-symmetric solitons, like the sphaleron[28, 12, 17, 18]. We study explicitly only the  $SU(2)$  case, and use the usual Witten[29] *Ansatz*, written in the form:

$$\begin{aligned} 2iA_i &= \epsilon_{iak}\sigma_a\hat{x}_k\left(\frac{\phi_1-1}{r}\right) \\ &- (\sigma_i - \hat{x}_i\hat{x}\cdot\sigma)\frac{\phi_2}{r} + \hat{x}_i\hat{x}\cdot\sigma H_1 \end{aligned} \quad (73)$$

$$U = \exp[i(\beta/2)\hat{x}\cdot\sigma] \quad (74)$$

where  $\hat{x}$  is the unit vector for  $\vec{x}$  and all functions in (73-74) depend only on the radial variable  $r$ . As is well-known there is a residual local  $U(1)$  gauge invariance under which  $\phi_1 + i\phi_2$  is multiplied by  $\exp(i\lambda)$ ,  $H_1$  transforms like a component of a  $U(1)$  Abelian gauge potential, and  $\beta \rightarrow \beta + \lambda(r)$ . We use this degree of freedom to choose  $\beta = \pi$ , which is the usual choice[28, 13, 12, 18] for the  $k = 0$  sphaleron. The equations of motion<sup>9</sup>are (primes denote  $d/dr$ ):

$$\begin{aligned} 0 &= (\phi_1' - H_1\phi_2)' + \frac{1}{r^2}\phi_1(1 - \phi_1^2 - \phi_2^2) \\ &+ (im - H_1)(\phi_2' + H_1\phi_1) - M^2(\phi_1 + 1) \end{aligned} \quad (75)$$

$$\begin{aligned} 0 &= (\phi_2' + H_1\phi_1)' + \frac{1}{r^2}\phi_2(1 - \phi_1^2 - \phi_2^2) \\ &- (im - H_1)(\phi_1' - H_1\phi_2) - M^2\phi_2 \end{aligned} \quad (76)$$

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<sup>9</sup>We correct some typographical sign mistakes in Refs.[17, 18] which did not appear in the equations these authors actually analyzed. It should also be noted that the analysis of the small- $r$  behavior of  $\phi_2$  and  $H_1$  given in Ref. [18] is not sufficiently general.



$$0 = \phi_1 \phi_2' - \phi_2 \phi_1' + H_1(\phi_1^2 + \phi_2^2) + \frac{im}{2}(1 - \phi_1^2 - \phi_2^2) + \frac{1}{2}M^2 r^2 H_1 \quad (77)$$

$$0 = \frac{1}{2}(r^2 H_1)' - \phi_2 \quad (78)$$

Equation (78) is the equation for  $U$ , and it is readily checked that it can be obtained by differentiating (77) and using (75-76). We will use (75-77) as a set of three independent equations.

The boundary conditions, for  $\beta = \pi$ , are:

$$\phi_1(r=0) = 1; \phi_1(\infty) = -1; \phi_2(0) = \phi_2(\infty) = 0. \quad (79)$$

The boundary conditions on  $H_1$  follow from (77), which is an algebraic equation for this quantity (see (81) below). Near  $r = \infty$  the approach to the values in (79) is exponential, while we find that  $\phi_2$  approaches zero linearly near  $r = 0$ .

These equations are complex, and have complex solutions, but with the change  $m \rightarrow im$  they would become real equations with real solutions. One can verify that it is consistent to choose  $\phi_1$  to be an even function of  $m$  and the other functions to be odd. Let us change to the dimensionless variables

$$H_1 = imA(r), \phi_2 = i(m/M)B(r), x = Mr; \quad (80)$$

from now on, primes denote  $d/dx$ . The equations of motion then become real, so we can choose  $A$  and  $B$  to be real also. A further simplification of notation is to drop the subscript on  $\phi_1$ , replacing it by  $\phi$ . Just as for the vortex soliton, the action, including the Chern-Simons term, is real.

Note that we can solve for  $H_1$  or  $A$  algebraically (as could D'Hoker and Vinet[17], although their equations differ from ours):

$$A = \frac{\phi' B - B' \phi - (1/2)(1 - \phi^2 + (m/M)^2 B^2)}{\phi^2 - (m/M)^2 B^2 + (1/2)x^2} \quad (81)$$

D'Hoker and Vinet [17] have a similar equation, except that the denominator is just  $\phi^2$  (they use a gauge with  $B = 0$  and have no mass  $M$ ). This creates singularities, since as we will see  $\phi$  always has at least one zero (exactly one for the  $m = 0$  sphaleron). The feedback of  $A$  in the D'Hoker-Vinet case leads

to infinitely many zeroes in  $\phi$  with an accumulation point at the origin, and  $\phi$  alternates between real and imaginary as it goes through its zeroes. The D'Hoker-Vinet soliton has infinite action (a logarithmic singularity). In our case, the mass term in the denominator of (81) is potentially stabilizing, and the  $B^2$  term is potentially destabilizing. However, for small  $m/M$  this term is too small to be harmful, since at small  $x$ ,  $\phi$  approaches unity. So there is a range of values of  $m/M$  where the D'Hoker-Vinet singularity is cured and finite-action solutions are expected.

We have studied these equations numerically, and find solutions for small values of  $m/M$ . As  $m/M$  increases it becomes increasingly difficult to find a numerical solution, so we have tried a simple variational approach with trial functions which are an excellent fit to the numerical solutions where we can find them. The variational approach indeed yields a minimum of the action for small  $m/M$  but for  $m/M \gtrsim 0.5$  the denominator in the equation (81) for  $A$  is singular or nearly so at some point and we can go no farther. Although we have no proof, it appears that there is no sphaleron-like solution for larger values of  $m/M$ . For small  $m/M$  the sphaleron solution is just a perturbation of the  $m = 0$  sphaleron[12, 13, 18]. We display in Figs. 2, 3, 4 the functions  $\phi$ ,  $A$ , and  $B$  as found from solving the unperturbed equation for  $\phi$  (the  $m = 0$  sphaleron), and the linearized equations for  $A$ ,  $B$  in this field  $\phi$ . The non-linear corrections to these functions are  $O(m^2/M^2)$ . Recall that the actual gauge potentials differ from  $A$  and  $B$  by factors  $\sim m$ , as in (80), so  $A$  and  $B$  go to zero linearly in this mass.

## 5 Conclusions

Although it is tempting to try, one cannot guess the fate of YMCS theory as  $k$  changes just from classical or semiclassical considerations. Once loop effects are considered, in a gauge-invariant way, one begins to see the same sort of tachyonic disease that is associated with pure  $d = 3$  YM theory (or with asymptotic freedom in  $d = 4$ ), if  $k \leq 29N/12$ . The cure for this disease is the generation of a dynamical mass  $M$  through quantum effects. The one-loop calculations we report in Section 2 are not quantitatively reliable, and this estimate of a critical value of  $k$  below which the theory is non-perturbative may well be off by a factor of two or so. This dynamical mass  $M$  is related in a self-consistent way to a condensate of quantum solitons

which is supported by configurational entropy. These effects may be crudely modeled by constructing a very approximate CJT one-dressed-loop action which suggests the way in which the condensate and mass  $M$  disappear above the critical value of  $k_c$ . Further support to these ideas comes from Section 4, where the solitons of the theory are examined. There are finite-action solitons in the small- $k$  condensate regime, but these either go away or become singular when the condensate disappears, supporting the notion of self-consistency between the existence of solitons and of a condensate.

We do not know of any other studies of the issues addressed here, and so there is nothing to compare with now. Certainly it would be interesting to make lattice-gauge simulations of YMCS theory, and we hope these are done. In order to make sense out of the approach of  $k$  to its critical value, one would like to have  $N$  large enough so that  $k/N$  behaves somewhat like a continuum parameter as  $k$  changes by unit values, but this is not easy for lattice simulations.

Let us recapitulate some of the estimates we have made:

1. For  $k \leq k_c$ , YMCS theory has a condensate, a dynamical mass, and a string tension. The critical value we estimate as

$$k_c/N \approx 2 \pm .7; \quad (82)$$

2. For  $k$  near but less than  $k_c$ , the dynamical mass  $M$  and condensate  $\langle \theta \rangle$  scale as:

$$M \sim (k_c - k)^{1/2}; \quad (83)$$

3. Under the same conditions, the string tension  $K_F$ , proportional to  $M^2$ , scales as:

$$K_F \sim k_c - k. \quad (84)$$

It would, of course, be interesting to study further variants on pure YMCS theory, e.g., adding fermions and scalars, and looking for non-perturbative dynamics in supersymmetric versions. Work in these directions is in progress.

## 6 Acknowledgements

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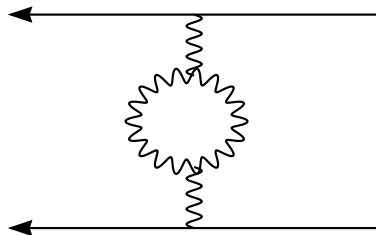
## Figure Captions

Fig. 1. One-loop  $S$ -matrix graphs from which the gauge-invariant propagator is extracted. Solid dots indicate a pinch vertex, as described in the text.

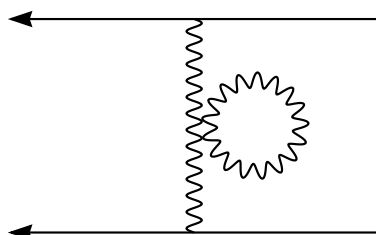
Fig. 2. The quantum soliton field  $\phi$ , for  $m = 0$ , plotted against  $x = Mr$ .

Fig. 3. The scaled quantum soliton field  $B$  (defined in the text). The actual field is found by multiplying by  $m/M$  when this quantity is small (higher-order corrections change  $B$  by  $O(m^2/M^2)$ ).

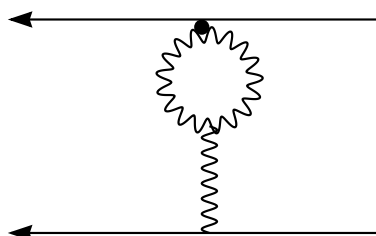
Fig. 4. The scaled quantum soliton field  $A$  (defined in the text). The actual field is found by multiplying by  $m$ , up to higher-order terms.



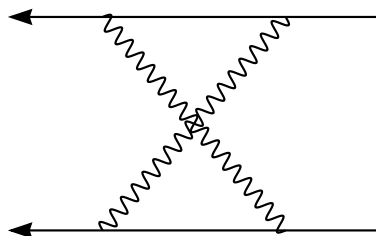
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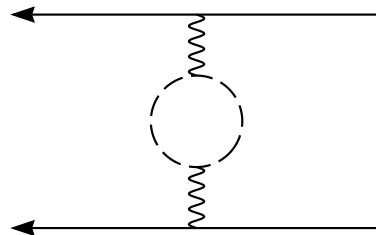
**c**



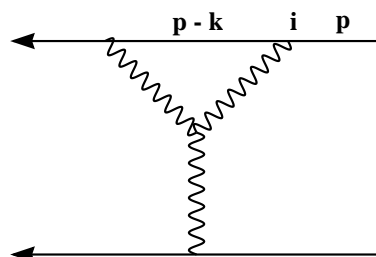
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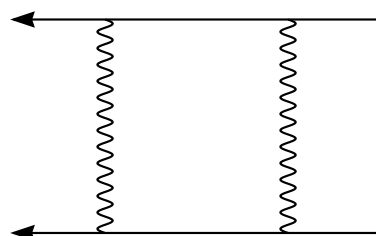
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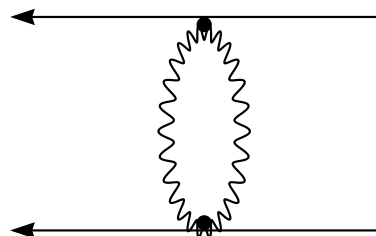
**b**



**d**



**f**



**h**



